

Friday, October 31, Lecture 26: Hilbert Spaces I

Last time...

- A function $f: \mathbb{R} \rightarrow \mathbb{C}$ is called periodic with period $2L$ if
$$f(x+2L) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

Equiv. to $f: [-L, L] \rightarrow \mathbb{C}$ with periodic b.c.s,
 $f(-L) = f(L)$ (or, f is defined on the circle)

- $f: [-\pi, \pi] \rightarrow \mathbb{C}$, $f(-\pi) = f(\pi)$.

- (Exponential) Fourier Series

$$T(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

What does this mean??
Let's find out.

- (Trigonometric) Fourier Series:

$$T(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

- Need "Linear algebra + convergence" to make sense of this.

Hilbert Spaces (aka Inner Product Spaces)

A Hilbert space $\mathcal{H}$ is a vector space with an inner product, $\langle \cdot, \cdot \rangle$.

- This induces a norm by

$$\|v\| := \sqrt{\langle v, v \rangle} \quad \text{for all } v \in \mathcal{H}.$$

3 Key examples:

• $\mathbb{C}^n$ with dot product. $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$.

$$\langle v, w \rangle = v \cdot w = \sum_{k=1}^n \bar{v}_k w_k = \bar{v}_1 w_1 + \bar{v}_2 w_2 + \dots + \bar{v}_n w_n$$

gives a norm $\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{|v_1|^2 + \dots + |v_n|^2}$

$$\bullet \ell^2(\mathbb{Z}) = \left\{ \begin{array}{l} \text{sequences } v: \mathbb{Z} \rightarrow \mathbb{C}, v = (\dots, v(-1), v(0), v(1), \dots) \\ \text{which are square summable:} \\ \|v\|^2 = \langle v, v \rangle < \infty \end{array} \right\},$$

where inner product is

$$\langle v, w \rangle = \sum_{k=-\infty}^{\infty} \bar{v}(k) w(k),$$

$\uparrow$ means finite, so sum converges.

$$\text{so } \|v\|^2 = \sum_{k=-\infty}^{\infty} \bar{v}(k) v(k) = \sum_{k=-\infty}^{\infty} |v(k)|^2.$$

$$\bullet L^2([- \pi, \pi]) = \left\{ \begin{array}{l} \text{functions } f: [- \pi, \pi] \rightarrow \mathbb{C} \\ \text{which are square integrable} \\ \|f\|^2 = \langle f, f \rangle < \infty \end{array} \right\}$$

↖ integral converges

where inner product is

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} g(x) dx, \text{ so } \|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx.$$

• Ex: $e^{ix} \in L^2([- \pi, \pi])$,

but $\frac{1}{\sqrt{|x|}} \notin L^2([- \pi, \pi])$, since $\int_0^{\pi} \left(\frac{1}{\sqrt{x}}\right)^2 dx = \int_0^{\pi} \frac{1}{x} dx = \log \pi - \lim_{x \rightarrow 0} \log x = \infty$.

• Inner products give us notion of angle:
in $\mathbb{R}^n$,



$$\langle v, w \rangle = \|v\| \|w\| \cos \theta$$

$$\text{so } \frac{\langle v, w \rangle}{\|v\| \|w\|} = \cos \theta$$

Think of this as defining angle!

Idea: use inner product to get geometric notion of angle!

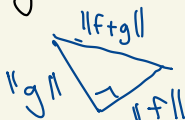
Still have the usual properties:

• (Triangle inequality) $\|f+g\| \leq \|f\| + \|g\|$

• (Cauchy-Schwarz) $\langle f, g \rangle \leq \|f\| \|g\|$

• (Pythagorean Thm) if $\langle f, g \rangle = 0$, then

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2$$



orthogonality!

So... what is orthogonality here?

Ex: in $\ell^2(\mathbb{Z})$, the vectors δ_k

$$\delta_n(k) = \begin{cases} 0 & n \neq k \\ 1 & n = k \end{cases}$$
$$\delta_k = (\dots, 0, \underset{\substack{\uparrow \\ k\text{th slot}}}{1}, 0, \dots)$$

Check: $\delta_n \in \ell^2(\mathbb{Z})$ since

$$\|\delta_n\|^2 = \sum_{k=-\infty}^{\infty} |\delta_n(k)|^2 = 1 < \infty.$$

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are orthonormal

$$\langle \delta_n, \delta_m \rangle = \sum_{k=-\infty}^{\infty} \overline{\delta_n(k)} \delta_m(k) = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

• Ex: in $L^2([-\pi, \pi])$, the functions from Fourier (HW) series are orthonormal:

$$\langle e^{inx}, e^{imx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} e^{imx} dx = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

So: what is an orthonormal basis? It should

be an orthonormal collection $\{v_k\}$ such that for any f ,

$$f = \sum_k \langle v_k, f \rangle v_k.$$

Ex: Let $\mathcal{U} = L^2([-\pi, \pi])$, and consider the collection

$$\{u_k\} = \{\sqrt{2}\sin x, \sqrt{2}\sin 2x, \dots, \sqrt{2}\sin kx, \dots\}$$

It's orthonormal, but look - if we choose $f(x) = x^2 \in \mathcal{U}$,

Then

$$\begin{aligned}\langle \sin kx, f \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{\sin kx} f(x) dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin kx \cdot x^2 dx \\ &= 0\end{aligned}$$

Since f is even and $\sin kx$ is odd.

So,

$$\sum_{k=1}^{\infty} 2\langle \sin kx, f \rangle \sin kx = 0$$

even though $f \neq 0$.

Three equivalent pictures:

- $f \in \mathcal{U}^{\perp}$, where $\{u_k\} \subseteq \mathcal{U}$
- $f \neq \sum_k \langle u_k, f \rangle u_k$
- $\sum_k |\langle u_k, f \rangle|^2 < \|f\|^2$

Def/Thm: orthonormal basis.

We call an orthonormal set $\{u_k\} \subseteq \mathcal{H}$ an orthonormal basis if any of the following equivalent conditions are met:

i) The orthogonal complement is trivial,
 $u^\perp = \{0\}$.

$$(or, \langle u_k, x \rangle = 0 \quad \forall k \Rightarrow x = 0)$$

$$ii) f = \sum_k \langle u_k, f \rangle u_k \quad \text{for all } f \in \mathcal{H}.$$

$$iii) \|f\|^2 = \sum_k |\langle u_k, f \rangle|^2 \quad \text{for all } f \in \mathcal{H}.$$

Ex: onb = orthonormal basis

• $\{\delta_k : k \in \mathbb{Z}\}$ is an onb of $\ell^2(\mathbb{Z})$

• $\{e^{ikx} : k \in \mathbb{Z}\}$ is an onb of $L^2([-\pi, \pi])$ ← Exponential Fourier series

• $\{\sin kx : k \in \mathbb{Z}\}$ is not an onb of $L^2([-\pi, \pi])$

because $\langle \cos x, \sin kx \rangle = 0$ so orthogonal complement is nontrivial. But

• $\{1\} \cup \{\sqrt{2} \cos kx : k \in \mathbb{Z}\} \cup \{\sqrt{2} \sin kx : k \in \mathbb{Z}\}$ is

← Trigonometric Fourier series

• Special Case:

Let $f \in L^2([-\pi, \pi])$.

Take the collection $\{v_k = e^{ikx} : k \in \mathbb{Z}\} \subset L^2([-\pi, \pi])$.

Then

$$\begin{aligned}\langle v_k, f \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{v_k} f \, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) \, dx \\ &=: \hat{f}(k)\end{aligned}$$

So

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}$$

This is the
exponential Fourier
series!

(Exercise: find the trigonometric Fourier coefficients)